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Transition to hyperchaos in chaotically forced coupled oscillators

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In this paper we discuss the Lyapunov and correlation dimensions of an attractor of a chain of unidirectionally coupled oscillators at the boundary of a chaos-hyperchaos transition. We discuss the problem of the distinction between chaos and hyperchaos based on these dimensions.

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Recently a growing interest in chaotically forced systems has been observed; for example, Refs. [1–4]. Chaotic forcing has some advantages in comparison with a periodic one. Using chaos we can, for example, correct nonlinear out-of-phase problems, eliminate fractal basin boundaries [3], and control unstable orbits [5]. The dynamics of chaotically forced systems is strictly connected with a chaotic signal-synchronization phenomenon [1,4,6–9]. Synchronization in chaotic systems seems to be interesting not only from the theoretical point of view. It gives rise to new applications, such as using chaos for secure communication [10,11]. As shown recently by de Sousa, Lichtenberg, and Lieberman [12], the boundary of the possible synchronization and nonsynchronization is strictly connected with the transition from chaotic to hyperchaotic behavior that is characterized by at least two positive Lyapunov exponents in a spectrum [13,14].

In this paper we investigate the properties of the Lyapunov and correlation dimensions of the attractor at the transition point from chaotic to hyperchaotic behavior. We discuss the problem of the distinction between chaos and hyperchaos based on these dimensions.

In our numerical investigation we have considered the dynamical behavior of a particular yet representative case, the chain of N unidirectionally coupled Duffing oscillators,

$$\frac{d^2x_1}{dt^2}+\delta\frac{dx_1}{dt}+x_1^3=\gamma\cos(\Omega t) , \tag{1a}$$

$$\frac{d^2x_n}{dt^2}+\beta\frac{dx_n}{dt}+x_n^3=x_{n-1} , \tag{1b}$$

where β, γ, δ, and Ω are constant, $n=1,2,\ldots,N$. In our system (1) the output from the preceding oscillator excites the next oscillator in the chain. We consider mainly the case when the response of the first oscillator is chaotic (chaotic forcing), but we also make some references to the case when this response is periodic (periodic forcing). In both cases we consider the Lyapunov exponent's spectrum of an attractor and Lyapunov dimension associated with it,

$$d_L=J+\frac{\sum_{i=1}^{j}\lambda_1}{|\lambda_{i+1}|} , \tag{2}$$

where j is determined by $\sum_{i=1}^{j}\geq 0$ but $\sum_{i=1}^{N+1}\lambda_i<0$. According to the Kaplan-Yorke conjecture [15], $d_L=d_I$, where d_I is the information dimension. Information and Lyapunov dimensions are related to the other attractor dimensions as follows:

$$d_c\geq d_I\approx d_L\geq d_{\rm corr} ,$$

where d_c is a capacity dimension, while $d_{\rm corr}$ is a correlation dimension [16]. Up till now there has been no effective way of estimating the capacity dimension of attractors of higher-dimensional systems, and this dimension will not be considered here. The correlation dimension, the second attractor dimension measure that we consider in this paper, is defined in terms of the scaling behavior of the so-called correlation integral. For a $(d=2N+1)$-dimensional embedding with trajectory vectors $\mathbf{x}_k$ we define the correlation sum as

$$C_d(R)=\sum_{k,l=1,k\neq l}^{M}\Theta(R-|x_k-x_l|)\ , \tag{3}$$

where M is the number of vectors in the data set being analyzed. Θ is the Heaviside step function. As is well known, the correlation integral $C_d(R)$ gives the average of the relative number of trajectory points within the distance R of another trajectory point. For many attractors, the correlation dimension is defined as that number that satisfies

$$d_{\text{corr}}=\frac{\log_{10}C(R)}{\log_{10}R} \tag{4}$$

in the scaling region. In practical situations the scaling region can be rather limited [17,18]. For larger R, larger than the size of the attractor, the correlation integral saturates at the value of unity. For small R, smaller than the smallest distance between data points, the integral goes to zero. Despite these inconveniences the correlation dimension has been found to be the simplest characterization of experimental attractors, for example [18,19], while estimation of Lyapunov exponents from time series can sometimes lead to incorrect results [20,21].

In the case of chaotic forcing we took $\delta=0.1$ (the famous Japanese attractor [22]) and considered β in the interval [0.1, 0.2]. The trajectories of Eq. (1a) are located on the three-dimensional (3D) manifold. If the trajectories of the whole system (1a) and (1b) are located in this 3D manifold as well, then the oscillators $n=2,\ldots,N$ simply reproduce the chaotic oscillations of the first oscillator as all trajectories converge to the attractor of Eq. (1a). The described manifold exists for any value of the coupled oscillators parameter β. This enables us to investigate the stability of the chaotic set located in this manifold as a function of β. The Lyapunov-exponent spectrum of the coupled systems (1a) and (1b) can be divided into two subsets $\lambda^{(1)}$ and $\lambda^{(2)}$, along and orthogonal to the manifold, respectively. The first subset of Lyapunov exponents is associated with driving system (1a) and consists of three exponents describing the evolution of perturbations tangent to the manifold. The Lyapunov exponents of the second subset describe the evolution of the perturbation transverse to the manifold. As shown recently by de Sousa, Lichtenberg, and Lieberman [12], they are equivalent to the conditional or sub-Lyapunov exponents of Pecora and Carroll [1]. The dependence of the Lyapunov dimension on β for a different number of oscillators is presented in Fig. 1, where we have also indicated the intervals where the Lyapunov exponents spectrum has one (dotted line), two (dashed line), three (solid line), and four (bold line) positive exponents. Lyapunov exponents have been computed using a computer software INSITE [23]. Computations have been performed with the β step equal to 0.005 in the whole interval [0.1, 0.2] and with a step 0.005 in the neighborhood of the chaos-hyperchaos transition. It clearly appears that for $n=2$ and 4 there are small regions for higher values of β where all $\lambda^{(2)}$-Lyapunov exponents are negative (β intervals: [0.173, 0.2] for $n=4$, [0.121, 0.2] for $n=2$]). In these intervals the chaotic limit set of the whole system (1a) and (1b) is located on the manifold of the attractor of the forcing system (1a). For the smaller values of β at least one $\lambda^{(2)}$-Lyapunov exponent is positive and the resulting limit set is not restricted to the manifold of the forcing system (1a), and we observe a hyperchaos regime. From Fig. 1 one can find that the Lyapunov dimension-control parameter β relation is a continuous function at the transition point from chaos to hyperchaos and at the transitions to higher levels of hyperchaos (when the new Lyapunov exponent becomes positive). Figures 2(a) and

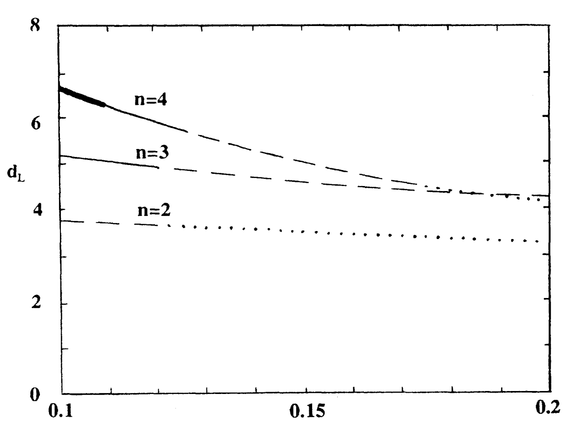

FIG. 1. Lyapunov dimension vs β plot for Eqs. (1): $\delta=0.1$, $\gamma=10.0$, $\Omega=1.0$ (chaotic forcing).

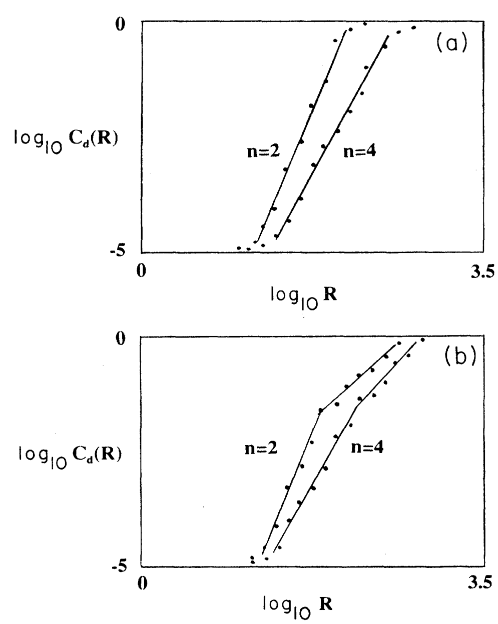

FIG. 2. Log-log plot of correlation integral vs distance R for Eqs. (1): $\delta=0.1$, $\gamma=10.0$, $\Omega=1.0$. (a) Chaotic regime $\beta=0.122$ for $n=2$ and $\beta=0.173$ for $n=4$, (b) hyperchaotic regime $\beta=0.120$ for $n=2$ and $\beta=0.171$ for $n=4$.

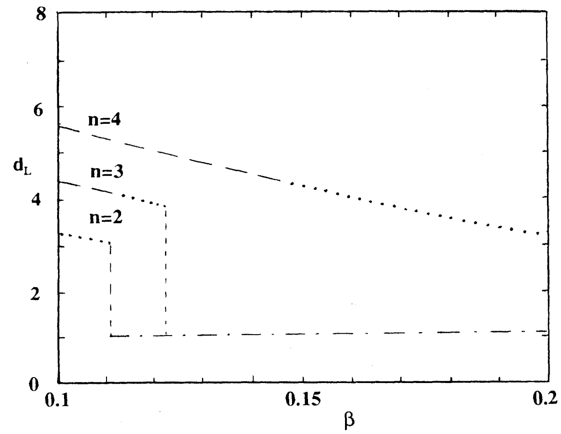

FIG. 3. Lypaunov dimension vs β plot for Eqs. (1): $\delta=0.2$, $\gamma=10.0$, $\Omega=1.0$ (periodic forcing).

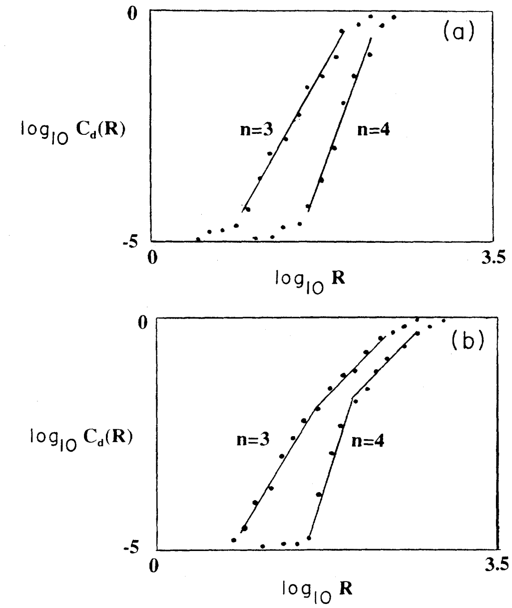

FIG. 4. Log-log plot of correlation integral vs distance R for Eqs. (1): $\delta=0.2$, $\gamma=10.0$, $\Omega=1.0$. (a) Chaotic regime $\beta=0.114$ for $n=3$ and $\beta=0.148$ for $n=4$, (b) hyperchaotic regime $\beta=0.148$ for $n=3$ and $\beta=0.145$ for $n=4$.

2(b) present log-log plots of the correlation integral (3) as a function of the distance R. In Fig. 2(a) the plot shows the results for the chaotic case (only one positive Lyapunov exponent) while in Fig. 2(b) we present a hyperchaotic case with two positive Lyapunov exponents. From these plots we can find that, in the case of chaos, the correlation integral exhibits only one scaling region while for the hyperchaotic case two scaling regions are visible. As shown before in the chaotic case, trajectories of all oscillators evolve on the 3D manifold of the first attractor, and their behavior is strongly connected (all oscillators evolve in the same region of phase space—on the attractor of the forcing oscillator). In the hyperchaotic case the other oscillators evolve in the larger-dimensional manifolds and their behavior is less connected with the behavior of the first one (forced oscillators do not evolve on the same attractor as a forcing oscillator). With this interpretation our results can be explained in terms of Lorenz conjectures [18].

To show the robustness of our conjecture that $d_L(\beta)$ is a continuous function and that the correlation dimension shifts from one to two scaling regions at the transition from chaotic to hyperchaotic behavior, we consider system (1a) and (1b) with periodic forcing. In this case we have taken $\delta=0.2$. The limit cycle of the first oscillator lies on the annulus, and of course all $\lambda^{(1)}$-Lyapunov exponents are nonpositive. If all of the $\lambda^{(2)}$-Lyapunov exponents are negative as well, then the periodic or quasiperiodic trajectories of the whole system also lie on this annulus. When one or more $\lambda^{(2)}$-exponents become positive we observe a chaotic or hyperchaotic regime. The dependence of the Lyapunov dimension on β in this case is presented in Fig. 3, where the number of positive Lyapunov exponents is indicated in the same way as in Fig. 1. It clearly appears that with the increase of β the behavior of the system becomes simpler, periodic for the chain of two and three oscillators, and chaotic (one positive Lyapunov exponent) for the chain of four oscillators. The transition from periodic to chaotic regimes is associated with the jump of the Lyapunov dimension, while the transition from chaotic to hyperchaotic behavior cannot be detected in a similar way, as the Lyapunov dimension is again a continuous function of β at this transition point. The log-log plots of the correlation integral (3) as a function of the distance R are shown in Fig. 4(a) (chaotic case) and Fig. 4(b) (hyperchaotic case). Again one can find the same property as that described for chaotic forcing: one scaling region in the chaotic regime and two scaling regions in the hyperchaotic regime.

To summarize it has been demonstrated here that the chain of chaotically forced coupled Duffing oscillators can show chaotic or hyperchaotic behavior. The same chain with periodic forcing can show periodic, chaotic, or hyperchaotic behavior. Chaotic and hyperchaotic regimes can be distinguished by the knowledge of the whole spectrum of Lyapunov exponents. Unfortunately this distinction cannot be made based on the Lyapunov dimension d_L or the associated information dimension $-d_I$, as the dependence of d_L on the system parameter β has been found to be a continuous function at the chaos-hyperchaos boundary for both periodic and chaotic forcing. This result is not trivial and quite surprising, as the information dimension of the attractor plays a crucial role in the experimental distinction between strange chaotic and strange nonchaotic attractors [20,21,24]. The correlation dimension allows us to follow up on the distinction between chaotic and hyperchaotic regimes as we observe a different number of scaling regions in both regimes. This property can be useful to follow up on the chaos-hyperchaos distinction based on a single experimental time series.

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